CS 357 D

Lecture 8

Orders and Lattices

http://cs357d.stanford.edu/

April 26, 2007

LATTICES AND ORDER

Supplementary Notes
based on

Introduction to Lattices and Order
by B.A. Davey and H.A. Priestley
Cambridge University Press, 2001

Order

Let P be a set. An order (or partial order) on P is a **binary relation** $\leq$ on P such that for all $x,y,z \in P$:

(i) $x \leq x$ (reflexivity)
(ii) $x \leq y$ and $y \leq z$ implies $x \leq z$ (transitivity)
(iii) $x \leq y$ and $y \leq x$ implies $x = y$ (antisymmetry)

The relation $\leq$ gives rise to the relation $<$ of strict inequality :

$x < y$ in P iff $x \leq y$ and $x \neq y$

Partially ordered set (Poset)

A set P equipped with an order relation $\leq$ is called a partially ordered set, or poset

Example: $(P , \leq)$ with

$P = \{ \bot , neg , zero , pos , \top \}$

$\leq = \{ (\bot , \bot) , (neg , neg) , (zero , zero) , (pos , pos) , (\top , \top) , (\bot , neg) , (\bot , zero) , (\bot , pos) , (\bot , \top) , (neg , \top) , (zero , \top) , (pos , \top) \}$

note that the elements neg , zero , and pos are not related to each other:
we don't have neg $\leq$ zero nor zero $\leq$ neg

Covering relation

(P , ≤) : ordered set $x,y \in P$

x is covered by y , written $x \lessdot y$ if

(i) $x < y$, and

(ii) $x \leq z < y$ implies $z = x$

intuitive meaning: there is nothing between x and y

For a finite set, the ordering relation is determined by the covering relation and v.v.

Covering relation

Examples:

$\{ \bot , neg , zero , pos , \top \}$

$\bot \lessdot neg$ $neg \lessdot \top$

$\bot \lessdot zero$ $zero \lessdot \top$

$\bot \lessdot pos$ $pos \lessdot \top$

$(N , \leq)$ $x \lessdot y$ if $y = x + 1$

$(\Re , \leq)$ no covering relation

$(\wp(X) , \subseteq)$ $A \lessdot B$ if $B = A \cup \{b\}$ for some $b \in X / A$

powerset of X

Hasse diagrams

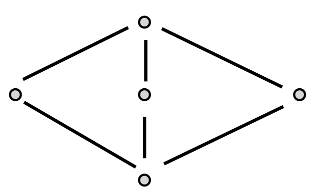
Hasse diagrams are a pictorial representation of the covering relation:

if $x \lessdot y$

x and y are connected by an edge, and x is drawn below y

Example:

covering relation for
$\{ \bot , neg , zero , pos , \top \}$



Special partially ordered sets

Chain

An ordered set P is a chain if for all $x, y \in P$ either $x \leq y$ or $y \leq x$ (all elements are comparable)

Also known as:
- linearly ordered set
- totally ordered set

Examples:

$(Z , \leq)$ (set of all integers with the standard order)

$(\{ \bot , \top \} , \{ (\bot , \bot) , (\top , \top) \})$

Special partially ordered sets

Antichain

An ordered set P is an antichain if for all $x, y \in P$
if $x \leq y$ then $x = y$ (no elements are comparable)

Examples:

$(Z , \{ (x,x) \mid x \in Z \})$ (set of all integers with reflexive relation)

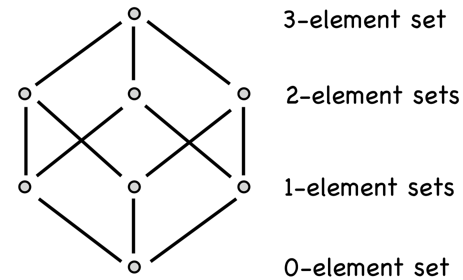
({ neg , zero , pos } ,
{ (neg , neg) , (zero , zero) , (pos , pos) })

Hasse diagrams -- Example

$\wp(\{1, 2, 3\}) =$

$\{ \emptyset , \{1\} , \{2\} , \{3\} , \{1,2\} , \{1,3\} , \{2,3\} , \{1,2,3\} \}$

Diagram of $(\wp(\{1,2,3\}) , \subseteq)$:



Maps between orders

$(P , \leq_P) , (Q , \leq_Q)$: ordered sets

$f : P \to Q$, function from P to Q , is

(i) monotone (order-preserving) if

$$x \leq_P y \text{ implies } f(x) \leq_Q f(y)$$

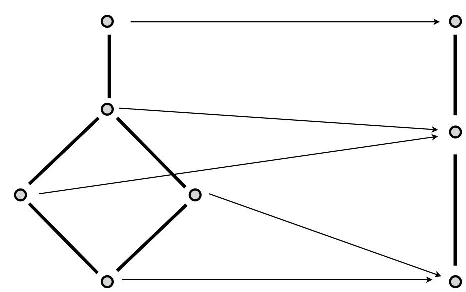
(ii) an order-embedding if

$$x \leq_P y \quad \text{iff} \quad f(x) \leq_Q f(y)$$

(iii) an order-isomorphism if

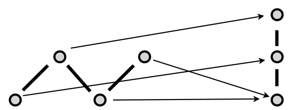
$$x \leq_P y \quad \text{iff} \quad f(x) \leq_Q f(y) \quad \text{and} \quad f \text{ is onto}$$

Maps between orders -- example



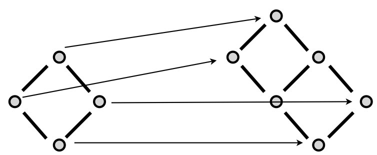
monotone, but not an order-embedding

Maps between orders -- example



monotone

not order-embedding



monotone

order-embedding

Top and Bottom

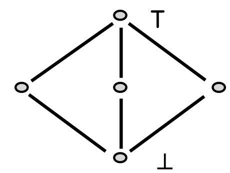
$(P, \leq_P)$: ordered set $\quad x \in P$

x is a bottom (least) element (⊥) of P if $\quad \forall y \in P \,.\, x \leq_P y$

x is a top (greatest) element (T) of P if $\quad \forall y \in P \,.\, y \leq_P x$

top and bottom may not exist

top and bottom are unique if they exists



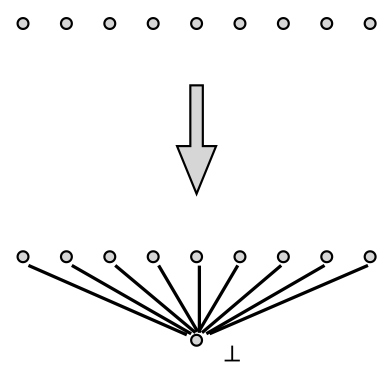
no top, no bottom

Top and bottom -- examples

$(\wp(X), \subseteq)$:	$\bot = \varnothing$	$T = X$
$(\{ n \mid n \geq 0 \}, \leq)$:	$\bot = 0$	no top
$(\{ x \in R \mid a \leq x \leq b \}, \leq)$:	$\bot = a$	$T = b$
$(\{ x \in R \mid a < x < b \}, \leq)$:	no bottom	no top

Lifting

Add a bottom element to an otherwise unordered set



S

$S_\bot = S \cup \{ \bot \}$

Maximal and minimal elements

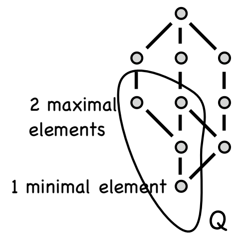
$(P, \leq_P)$: ordered set, $Q \subseteq P$, $x \in Q$

x is a maximal element of Q if

for all $y \in Q$: $x \leq y$ implies $x = y$

x is a minimal element of Q if

for all $y \in Q$: $y \leq x$ implies $x = y$



Example:

$(\wp(N)\backslash N, \subseteq)$ has maximal elements $N\backslash\{n\}$ for all $n \in N$

Down-sets and Up-sets

$(P, \leq_P)$: ordered set, $Q \subseteq P$

Q is an **up-set** (order filter , increasing set) if

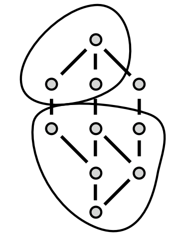
for all $x \in Q$, $y \in P$: $x \leq y$ implies $y \in Q$

(Q is closed under going up)

Q is a **down-set** (order ideal , decreasing set) if

for all $x \in Q$, $y \in P$: $y \leq x$ implies $y \in Q$

(Q is closed under going down)



$\uparrow Q = \{ y \in P \mid \exists x \in Q \,.\, x \leq y \}$ smallest up-set that includes Q

$\downarrow Q = \{ y \in P \mid \exists x \in Q \,.\, y \leq x \}$ smallest down-set that includes Q

Upper bound

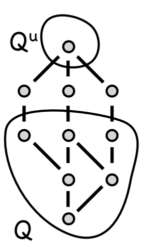
$(P, \leq_P)$: ordered set, $Q \subseteq P$, $x \in P$

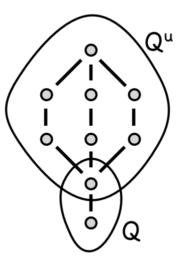
x is an **upper bound** of Q if $\forall y \in Q \,.\, y \leq_P x$

$Q^u : \{ x \in P \mid \forall y \in Q \,.\, y \leq_P x \}$ all upper bounds of Q

if Q^u has a least element x:

x is called the **least upper bound (lub)** or **supremum (sup)** of Q





if Q has a top element T: sup Q = T

Lower bound

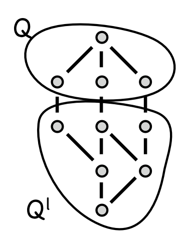
$(P, \leq_P)$: ordered set, $Q \subseteq P$, $x \in P$

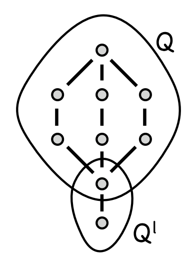
x is a **lower bound** of Q if $\forall y \in Q \,.\, x \leq_P y$

$Q^l : \{ x \in P \mid \forall y \in Q \,.\, x \leq_P y \}$ all lower bounds of Q

if Q^l has a greatest element x:

x is called the **greatest lower bound (glb)** or **infimum (inf)** of Q





if Q has a bottom element $\bot$: inf Q = $\bot$

Upper and lower bounds

$(P, \leq_P)$: ordered set, assume P has $\bot$ and $\top$

$\sup P = \top$ $\qquad \inf P = \bot$

$\sup \varnothing = \bot$ $\qquad \inf \varnothing = \top$ $\qquad$ Note: $\varnothing^u = \varnothing^l = P$

Notation:

$\sup(\{x, y\}) = x \vee y$ $\qquad$ join $\qquad \sup Q = \bigvee Q$

$\inf(\{x, y\}) = x \wedge y$ $\qquad$ meet $\qquad \inf Q = \bigwedge Q$

Some properties:

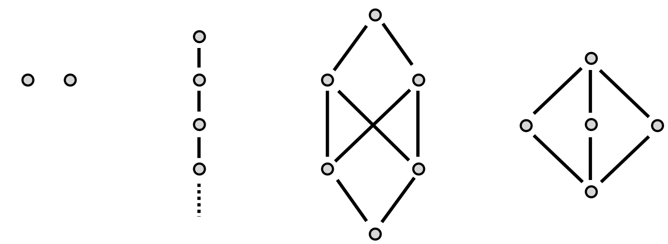
if $x \leq_P y$ then $x \wedge y = x$ and $x \vee y = y$

Lattice

$(P, \leq_P)$: ordered set

P is a lattice if $x \wedge y$ and $x \vee y$ exist for all $x, y \in P$

P is a complete lattice if $\bigvee Q$ and $\bigwedge Q$ exist for all $Q \subseteq P$



not a lattice | lattice not complete | not a lattice | complete lattice

Lattice

Note:

$\varnothing \subseteq P$, $P \subseteq P$, so in a complete lattice

$\bigvee \varnothing$, $\bigwedge \varnothing$, $\bigvee P$ and $\bigwedge P$ must all exist

☛ complete lattice must be bounded

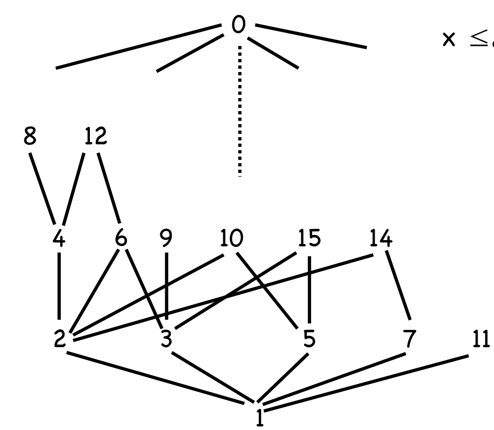
Examples:

$(\wp(X), \subseteq)$ is a complete lattice for any set X

$(N, \leq_{div})$ is a complete lattice

divides relation

Complete lattice -- example



$x \leq_{div} y$ if x divides y

$x \wedge y = \gcd(x, y)$

$x \vee y = \mathrm{lcm}(x, y)$

$\top_{div} = 0$

$\bot_{div} = 1$

Lattice as algebraic structure

$(P, \leq_P)$: lattice

Define $\wedge$ and $\vee$ as functions: $\wedge, \vee : P^2 \rightarrow P$

$x \wedge y = \inf\{x, y\}$

$x \vee y = \sup\{x, y\}$

Properties: $\wedge, \vee$ are order preserving

$x \leq_P z$ implies $x \wedge y \leq_P z \wedge y$ and $x \vee y \leq_P z \vee y$

$x \leq_P z$ and $y \leq_P t$ implies $x \wedge y \leq_P z \wedge t$ and

$x \vee y \leq_P z \vee t$

Lattice as algebraic structure

Properties:

associative: $(x \vee y) \vee z = x \vee (y \vee z)$ $(x \wedge y) \wedge z = x \wedge (y \wedge z)$

commutative: $x \wedge y = x \wedge y$ $x \vee y = x \vee y$

idempotent: $x \vee x = x$ $x \wedge x = x$

absorption $x \wedge (x \vee y) = x$ $x \vee (x \wedge y) = x$

In a lattice, join and meet are determined by the order and v.v.

$(P, \leq_P)$ $(P, \vee_P, \wedge_P)$ can be used interchangeably

Lattice as algebraic structure

Properties:

$\bot \wedge a = \bot$ $\bot \vee a = a$ $\bot$ acts like 0

$T \wedge a = a$ $T \vee a = T$ T acts like 1

Alternative representations: $(P, \leq_P)$ $(P, \wedge, \vee)$

Note:

in (N , lcm , gcd): 0 = 1 and 1 = 0

Homomorphism

$(P, \leq_P), (Q, \leq_Q)$: ordered sets

$f : P \rightarrow Q$ is a lattice homomorphism if

$f(x \vee_P y) = f(x) \vee_Q f(y)$ join preserving

$f(x \wedge_P y) = f(x) \wedge_Q f(y)$ meet preserving

f is a lattice homomorphism iff it is an order isomorphism

Knaster-Tarski fixed point theorem

$(P, \leq_P)$: complete lattice

$f : P \to P$ order preserving function (monotone)

greatest fixed point of f : $\text{gfp}(f) = \bigwedge \{ x \in P \mid x \leq_P f(x) \}$

least fixed point of f : $\text{lfp}(f) = \bigwedge \{ x \in P \mid f(x) \leq_P x \}$

if f is **increasing** $(x \leq_P f(x))$ **lfp(f)** can be obtained by starting with **⊥** and repeatedly applying f :

$$\bot_P \longrightarrow f(\bot_P) \longrightarrow f(f(\bot_P)) \longrightarrow f(f(f(\bot_P))) \longrightarrow \cdots$$

until a fixed point is reached: $f^n(\bot_P) = f^{n+1}(\bot_P)$

Knaster-Tarski fixed point theorem

$(P, \leq_P)$: complete lattice

$f : P \to P$ order preserving function (monotone)

greatest fixed point of f : $\text{gfp}(f) = \bigwedge \{ x \in P \mid x \leq_P f(x) \}$

least fixed point of f : $\text{lfp}(f) = \bigwedge \{ x \in P \mid f(x) \leq_P x \}$

if f is **decreasing** $(f(x) \leq_P x)$ **gfp(f)** can be obtained by starting with **T** and repeatedly applying f :

$$\top_P \longrightarrow f(\top_P) \longrightarrow f(f(\top_P)) \longrightarrow f(f(f(\top_P))) \longrightarrow \cdots$$

until a fixed point is reached: $f^n(\top_P) = f^{n+1}(\top_P)$

Ascending and descending chain condition

$(P, \leq_P)$: ordered set

(i) **P has length n** if the longest chain in P has length n

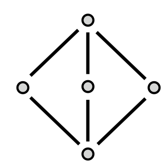
(ii) **P has finite length** if all its chains are finite

(iii) P satisfies the **ascending chain condition (ACC)** if for any sequence $x_1 \leq_P x_2 \leq_P$ ...... of elements in P there exists k such that $x_k = x_{k+1} =$ ..... .

(iv) P satisfies the **descending chain condition (DCC)** if for any sequence $x_1 \geq_P x_2 \geq_P$ .... of elements in P there exists k such that $x_k = x_{k+1} =$ ....

Note: A finite lattice satisfies both the ACC and the DCC

Ascending and descending chain condition: examples



length 3
satisfies both ACC and DCC

$(N, \leq)$ infinite chain
satisfies DCC, but not ACC

$(\wp(X), \subseteq)$ for finite set X of size n:
length n+1
satisfies both ACC and DCC